

2D Lattice Model Construction of Symmetry Protected Topological Phases

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We propose a general approach to construct symmetry protected topological (SPT) states (*ie* the short-range entangled states with symmetry) in 2D spin/boson systems on lattice. In our approach, we fractionalize spins/bosons into different fermions, which occupy nontrivial Chern bands. After the Gutzwiller projection of the free fermion state obtained by filling the Chern bands, we can obtain SPT states on lattice. In particular, we constructed a U(1) SPT state, a SO(3) SPT state, and a SU(2) SPT state on lattice.

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I. INTRODUCTION

Recently, quantum entanglement becomes an important concept and a powerful tool in the fields of quantum information and quantum many-body systems.^{1,2} A long-range entangled state (LRE) can not be deformed to a direct product state by any local unitary (LU) transformation.^{3,4} The fractional quantum Hall effect (FQHE)⁵ is a concrete example of LRE and has been considered as an example of intrinsic “topological order”.⁶⁻⁸ If a gapped quantum state is short-range entangled (SRE), one can always deform it to a spatially direct product state by a LU transformation³. If a SRE state has no symmetry-breaking and is invariant under some symmetry group G , and if there is no way to locally deform the state to direct product state unless one explicitly breaks the symmetry in LU procedure, we call such SRE states “Symmetry-Protected Topological Phases” (SPT). Topological insulators⁹ (TI) belongs to fermionic SPT protected by \mathbb{Z}_2 time reversal symmetry and U(1) particle number conservation. On the other hand, a gapped quantum state is long-range entanglements and symmetry will be called “Symmetry-Enriched Topological Phases” (SET).

Group theory has been successfully utilized in analyzing symmetry-breaking phases. Recently, it was shown that group cohomology with U(1) coefficient can be utilized in classification and construction of SPT phases.³ A concrete SPT phase is the 1D Haldane phase with SO(3) symmetry where integer spin is a faithful representation of SO(3) symmetry. It is known that the spin-1 antiferromagnetic chain is semiclassically (in large spin- S expansion) described by a nonlinear sigma model (NLSM) with 2π topological theta term. The two boundaries are occupied by two dangling free 1/2 spins which are protected by SO(3) symmetry and contributes to fourfold degeneracy.

It is interesting that, by using group cohomology theory, one can construct exactly solvable lattice models which realize specific SPT phases respecting given symmetry groups^{3,10,11}. Along this line, some interesting lattice models of bosonic SPT phases with discrete symme-

try groups have been proposed in literatures. For continuous symmetry group, however, the solution of cocycle construction in group cohomology is more difficult mathematically. On the one hand, Liu and Wen¹² study 2D SPT phases with SO(3) and with SU(2) symmetries and relate the edge of SPT phases to Wess-Zumino-Witten (WZW) field theory¹³⁻¹⁵. On the other hand, a concrete lattice model for realizing SPT phases is much desired, which is more physical than continuum field theory description.

Instead of directly constructing exactly solvable models, in this paper, we present an effective approaches to realizing SPT phases on 2D lattice. Our approach is based on the projective construction of strongly correlated bosonic or spin systems,¹⁶⁻²⁷ in which bosons/spins are fractionalized into several fermions, which occupy several independent nontrivial Chern bands respectively. After the Gutzwiller projection of the free fermion state obtained by filling the Chern bands, we can obtain SPT states on lattice. In this paper, we have constructed a U(1) SPT state, a SO(3) SPT state, and a SU(2) SPT state using the projective construction.

II. U(1) SPT STATES IN SPIN-1 LATTICE MODEL

In this section, we are going to construct a SPT phase with U(1) symmetry on lattice.^{3,28-30} Such a state has been obtained by several other constructions.³¹⁻³³ Our lattice model is a spin-1 model on a honeycomb lattice, with 3 states, $|m\rangle$, $m = 0, \pm 1$, on each site. We can view our spin-1 model as spin-1/2 hardcore-boson model with $|m = 0\rangle$ state as the no-boson state and $|m = \pm 1\rangle$ state as the one-boson state with spin-up or spin-down.

In a fermionic projective construction, we write the bosonic operator as

$$b_{\alpha,i} = f_{\alpha,i} c_i, \quad \alpha = \uparrow, \downarrow \quad (1)$$

and we regard the 3 states, $|m = 0, \pm 1\rangle$, on each site as

states in a fermion system described by $f_{i,\alpha}$ and c_i :

$$|m=0\rangle = |0\rangle, \quad |m=1\rangle = f_{\uparrow}^{\dagger} c^{\dagger} |0\rangle, \quad |m=-1\rangle = f_{\downarrow}^{\dagger} c^{\dagger} |0\rangle \quad (2)$$

We note that the above three physical states are the only states in the fermion system that f -fermion number and c -fermion number are equal on each site. Therefore, we can start with a many-fermion state of $f_{i,\alpha}$ and c_i , $|\Psi\rangle$, and obtain a physical spin state $|\Phi\rangle$ by projecting into the sub-space with equal f -fermions and c -fermions on each site:

$$|\Phi\rangle = P|\Psi\rangle. \quad (3)$$

Using such a projective construction, we would like to construct an $U(1)$ SPT state on a honeycomb lattice.

Let us consider the following free fermion Hamiltonian

$$H_{MF} = \sum_{i,j} \left[f_j^{\dagger} U_{ji} f_i + c_j^{\dagger} v_{ji} c_i \right], \quad (4)$$

where U_{ji} are 2×2 matrices satisfying

$$U_{ji}^{\dagger} = U_{ij}, \quad (5)$$

and v_{ji} complex numbers satisfying

$$v_{ji}^* = v_{ij}, \quad (6)$$

Let $|\Psi\rangle$ be the lowest energy state of the above free fermion Hamiltonian. Then $|\Phi\rangle = P|\Psi\rangle$ will be a spin/boson state induced by the above free fermion Hamiltonian:

$$\Phi(\{m_i\}) = \langle 0 | \prod_i \mathcal{O}_i | \Psi \rangle, \quad (7)$$

where, $m_i = 0, \pm 1$, $\{m_i\}$ defines a complete Ising basis of Hilbert space of original spin-1 model. The operator \mathcal{O}_i is defined as follows. If $m_i = 0$, $\mathcal{O}_i = \mathbb{I}$ where \mathbb{I} is identity operator; if $m_i = 1$, $\mathcal{O}_i = f_{i\uparrow} c_i$; if $m_i = -1$, $\mathcal{O}_i = f_{i\downarrow} c_i$. It is easy to check that before projection, the Hilbert space dimension of each site is 8 (spanned by $|0\rangle, f_{\sigma}^{\dagger} |0\rangle, c^{\dagger} |0\rangle, f_{\uparrow}^{\dagger} f_{\downarrow}^{\dagger} |0\rangle, f_{\uparrow}^{\dagger} c^{\dagger} |0\rangle, f_{\downarrow}^{\dagger} c^{\dagger} |0\rangle, \sigma = \uparrow, \downarrow$). After projection, the physical spin Hilbert space at each site is reduced to 3 (spanned by $|0\rangle, f_{\sigma}^{\dagger} c^{\dagger} |0\rangle$). We say that such a spin/boson state is described by the ansatz U_{ji} and v_{ji} .

To construct a SPT state using the above projective construction, we choose the ansatz to be³⁴

$$U_{ij} = \sigma^0, \quad ij = \text{nearest neighbour links} \quad (8)$$

$$U_{ij} = i\nu_{ij} t \sigma^3, \quad ij = \text{next nearest neighbour links},$$

and

$$v_{ij} = 1, \quad ij = \text{nearest neighbour links} \quad (9)$$

$$v_{ij} = i\nu_{ij} t, \quad ij = \text{next nearest neighbour links},$$

where $\nu_{ij} = +$ if the fermion f makes a right turn going from j to i on the honeycomb lattice, and $\nu_{ij} = -$ if the fermion f makes a left turn. For such an ansatz, the fermion band structure contains two bands with a gap at zero energy. The lower band has a Chern number +1 for the f_{\uparrow} -fermions and c -fermions and a Chern number -1 for the f_{\downarrow} -fermions. This ansatz determines the fermionic wavefunction, *ie* the unprojected state $|\Psi\rangle$. In the following, we will show that $|\Phi\rangle = P|\Psi\rangle$ will be a bosonic SPT state protected by the $U(1)$ symmetry (generated by S^z). First, from the free fermion Hamiltonian, we see that the ground state state $|\Psi\rangle$ respects the S_z spin rotation symmetry generated by $f_{\uparrow}^{\dagger} f_{\uparrow} - f_{\downarrow}^{\dagger} f_{\downarrow}$. The f_{α} -fermions form a “spin Hall” state described by $U(1) \times U(1)$ Chern-Simons theory

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} (a_{\uparrow\mu} \partial_{\nu} a_{\uparrow\lambda} \epsilon^{\mu\nu\lambda} - a_{\downarrow\mu} \partial_{\nu} a_{\downarrow\lambda} \epsilon^{\mu\nu\lambda}) \\ &\quad + \frac{1}{2\pi} A_{\mu}^{\text{spin}} \partial_{\nu} (a_{\uparrow\lambda} - a_{\downarrow\lambda}) \epsilon^{\mu\nu\lambda} \\ &= \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_{\nu} a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_{\mu}^{\text{spin}} \partial_{\nu} a_{I\lambda} \epsilon^{\mu\nu\lambda} \end{aligned} \quad (10)$$

with $I, J = 1, 2$ and

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (11)$$

Here A_{μ}^{spin} is the gauge potential that couple to the S_z spin density and current. The c -fermions form a “integer quantum Hall” state described by $U(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} b_{\mu} \partial_{\nu} b_{\lambda} \epsilon^{\mu\nu\lambda} \quad (12)$$

Thus the total effective theory is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_{\nu} a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_{\mu}^{\text{spin}} \partial_{\nu} a_{I\lambda} \epsilon^{\mu\nu\lambda} \\ &\quad + \frac{1}{4\pi} b_{\mu} \partial_{\nu} b_{\lambda} \epsilon^{\mu\nu\lambda} \end{aligned} \quad (13)$$

The projection P is done by setting the total f_{α} -fermion density-current, $J_{\mu}^f = \frac{1}{2\pi} \partial_{\nu} (a_{\uparrow\lambda} + a_{\downarrow\lambda}) \epsilon^{\mu\nu\lambda}$, equal to the c -fermion density-current, $J_{\mu}^c = \frac{1}{2\pi} \partial_{\nu} b_{\lambda} \epsilon^{\mu\nu\lambda}$.^{35,36} After setting $b_{\lambda} = (a_{\uparrow\lambda} + a_{\downarrow\lambda})$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi} \bar{K}_{IJ} a_{I\mu} \partial_{\nu} a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_{\mu}^{\text{spin}} \partial_{\nu} a_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (14)$$

with

$$\bar{K} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{K}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \quad (15)$$

Eqn. (14) is the low energy effective theory for the spin/boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. Such a low energy effective theory describes a $U(1)$ SPT state with an integer Hall conductance

$$\sigma_{xy} = q^T \bar{K}^{-1} q \frac{1}{2\pi} = -4 \frac{1}{2\pi}. \quad (16)$$

We note that the Hall conductance is an even integer in the unit of $\frac{1}{2\pi}$.

III. SO(3) SPT STATES IN A SPIN-1 BOSON MODEL ON LATTICE

In this section, we are going to construct a SPT phase protected by $SO(3)$ symmetry on lattice. Our lattice model contain spin-1 bosons and three spin-0 bosons on a honeycomb lattice. In a fermionic projective construction, we write the spin-1 bosonic operator as

$$b_{m,i} = f_{m,i}c_i, \quad m = 0, \pm 1 \quad (17)$$

and the three spin-0 bosonic operator as

$$\tilde{b}_{m,i} = \tilde{f}_{m,i}c_i, \quad m = 0, \pm 1 \quad (18)$$

Again, we can start with a many-fermion state of $f_{m,i}$, $\tilde{f}_{m,i}$ and c_i , $|\Psi\rangle$, and obtain a physical spin state $|\Phi\rangle$ by projecting into the sub-space where the sum of the f -fermion number and the \tilde{f} -fermion number is equal to the c -fermion number on each site:

$$|\Phi\rangle = P|\Psi\rangle. \quad (19)$$

Using such a projective construction, we can construct an $SO(3)$ SPT state on honeycomb lattice.

Let us consider the following free fermion Hamiltonian

$$H = \sum_{i,j} \left[f_j^\dagger u_{ji} f_i + \tilde{f}_j^\dagger u_{ji}^* \tilde{f}_i + c_j^\dagger u_{ji} c_i \right] \quad (20)$$

where u_{ji} are complex numbers satisfying

$$u_{ji}^* = u_{ij}, \quad (21)$$

To construct a $SO(3)$ SPT state using the above projective construction, we choose the ansatz to be

$$\begin{aligned} u_{ij} &= 1, \quad ij = \text{nearest neighbour links} \\ u_{ij} &= i\nu_{ij}t, \quad ij = \text{next nearest neighbour links,} \end{aligned} \quad (22)$$

where $\nu_{ij} = +$ if the fermion f makes a right turn going from j to i on the honeycomb lattice, and $\nu_{ij} = -$ if the fermion f makes a left turn.

For such an ansatz, the fermion band structure contains two bands with a gap at zero energy. The lower band has a Chern number +1 for the f -fermions and c -fermions and a Chern number -1 for the \tilde{f} -fermions. Let $|\Psi\rangle$ be the ground state of the above free fermion Hamiltonian where all the lower bands are filled with f -fermions, \tilde{f} -fermions and c -fermions. In the following, we like to show that $|\Phi\rangle = P|\Psi\rangle$ will be a bosonic SPT state protected by the $SO(3)$ symmetry (generated by $f^\dagger T^l f$, where T^l , $l = 1, 2, 3$, are the 3×3 matrices that generate the $SO(3)$ Lie algebra).

First, from the free fermion Hamiltonian, we see that the ground state state $|\Psi\rangle$ respects the $SO(3)$ spin rotation symmetry generated by $f^\dagger T^l f$. The f -fermions form an “integer quantum Hall” state described by $U^3(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu^{S_z} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (23)$$

with $I, J = 1, 2, 3$ and

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \quad (24)$$

Here $A_\mu^{S_z}$ is the gauge potential that couple to the S_z spin density and current. The \tilde{f} -fermions also form an “integer quantum Hall” state with an opposite Hall conductance, which is described by $U^3(1)$ Chern-Simons theory

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda} \quad (25)$$

The c -fermions form an “integer quantum Hall” state described by $U(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} b_\mu \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda} \quad (26)$$

Thus the total effective theory is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} - \frac{1}{4\pi} K_{IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda} \\ &\quad + \frac{1}{2\pi} q_I A_\mu^{S_z} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{4\pi} b_\mu \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda} \end{aligned} \quad (27)$$

The projection P is done by setting the total f -fermion and \tilde{f} -fermion density-current, $J_\mu^f = \sum_I \frac{1}{2\pi} \partial_\nu (a_{I\lambda} + \tilde{a}_{I\lambda}) \epsilon^{\mu\nu\lambda}$, equal to the c -fermion density-current, $J_\mu^c = \frac{1}{2\pi} \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda}$. After setting $b_\lambda = \sum_I (a_{I\lambda} + \tilde{a}_{I\lambda})$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi} \bar{K}_{IJ} \bar{a}_{I\mu} \partial_\nu \bar{a}_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} \bar{q}_I A_\mu^{S_z} \partial_\nu \bar{a}_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (28)$$

with $I, J = 1, 2, 3, 4, 5, 6$ and

$$\bar{K} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (29)$$

Also $\bar{a}_{I,\mu} = a_{I\mu}$ and $\bar{a}_{I+3,\mu} = \tilde{a}_{I\mu}$, $I = 1, 2, 3$. Eqn. (28) is the low energy effective theory for the spin-1 boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. Such a low energy effective theory describes a $SO(3)$ SPT state with an integer Hall conductance for the S_z “charge”

$$\sigma_{xy} = \bar{q}^T \bar{K}^{-1} \bar{q} \frac{1}{2\pi} = 2 \frac{1}{2\pi}. \quad (30)$$

Again, we note that the Hall conductance is an even integer in the unit of $\frac{1}{2\pi}$.

IV. SU(2) SPT STATE IN A SPIN-1/2 BOSON MODEL ON LATTICE

SU(2) SPT phases were firstly studied in Ref.12 where field theoretic approach is applied. The bulk is described by SU(2) principal chiral nonlinear sigma model

(the field variable g is 2×2 SU(2) matrix) with $2\pi K$ topological theta term where integer K labels different SPT phases. The boundary is described by a SU(2) $_K$ WZW term where SU(2) transformation is defined as left-multiplication, $ie\ g \rightarrow hg$ with $h \in \text{SU}(2)_L$, such that only left mover carries SU(2) charge and right mover is SU(2) charge-neutral. Due to this chiral SU(2) transformation, the boundary can not be simply replaced by a SU(2) $_K$ WZW critical spin chain that is nonchiral.

In the following, we are going to construct a SPT phase with a SU(2) symmetry on lattice. Our lattice model contains spin-1/2 bosons. Using the fermionic projective construction, we write the spin-1/2 bosonic operator as

$$b_{\alpha,i} = f_{\alpha,i} c_i, \quad \alpha = \uparrow, \downarrow \quad (31)$$

Again, we can start with a many-fermion state of $f_{\alpha,i}$ and c_i , $|\Psi\rangle$, and obtain a physical spin state $|\Phi\rangle$ by projecting into the sub-space where the f -fermion number is equal to the c -fermion number on each site:

$$|\Phi\rangle = P|\Psi\rangle. \quad (32)$$

Using such a projective construction, we can construct an SU(2) SPT state on lattice.

Let us consider the following free fermion Hamiltonian

$$H = \sum_{i,j} \left[f_{\alpha j}^\dagger u_{ji} f_{\alpha i} + c_j^\dagger v_{ji} c_i \right] \quad (33)$$

where u_{ji} and v_{ji} are complex numbers satisfying

$$u_{ji}^* = u_{ij}, \quad v_{ji}^* = v_{ij}. \quad (34)$$

To construct a SU(2) SPT state using the above projective construction, we choose the ansatz u_{ji} and v_{ji} such that the ground state of the above free fermion Hamiltonian is formed by f -fermions filling a Chern number +1 band and c -fermions filling a Chern number -2 band. In literature, there have been many concrete lattice models of realizing the band structure with nontrivial Chern number, such as Refs. 37–41. Recently the method of constructing a tight-binding model with arbitrary Chern number is proposed⁴⁰. Let $|\Psi\rangle$ be such a free fermion ground state. In the following, we like to show that $|\Phi\rangle = P|\Psi\rangle$ will be a bosonic SPT state with the SU(2) symmetry.

First, from the free fermion Hamiltonian, we see that the ground state $|\Psi\rangle$ respects the SU(2) spin rotation symmetry generated by $f^\dagger \sigma^l f$. The f -fermions form an “integer quantum Hall” state described by $U^2(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu^{S_z} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (35)$$

with $I, J = 1, 2$ and

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}. \quad (36)$$

Here $A_\mu^{S_z}$ is the gauge potential that couples to the S_z spin density and current. The c -fermions form an “integer quantum Hall” state described by $U^2(1)$ Chern-Simons theory

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} b_{I\mu} \partial_\nu b_{J\lambda} \epsilon^{\mu\nu\lambda} \quad (37)$$

Thus the total effective theory is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} - \frac{1}{4\pi} K_{IJ} b_{I\mu} \partial_\nu b_{J\lambda} \epsilon^{\mu\nu\lambda} \\ & + \frac{1}{2\pi} q_I A_\mu^{S_z} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \end{aligned} \quad (38)$$

The projection P is done by setting the total f -fermion density-current, $J_\mu^f = \sum_I \frac{1}{2\pi} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda}$, equal to the c -fermion density-current, $J_\mu^c = \sum_I \frac{1}{2\pi} \partial_\nu b_{I\lambda} \epsilon^{\mu\nu\lambda}$. After setting $b_{2\lambda} = -b_{1\lambda} + \sum_I a_{I\lambda}$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi} \bar{K}_{IJ} \bar{a}_{I\mu} \partial_\nu \bar{a}_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} \bar{q}_I A_\mu^{\text{spin}} \partial_\nu \bar{a}_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (39)$$

with $I, J = 1, 2, 3$ and

$$\bar{K} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}. \quad (40)$$

Also $\bar{a}_{I,\mu} = a_{I\mu}$ $I = 1, 2$, and $\bar{a}_{3,\mu} = b_{1\mu}$. Using an invertible integer matrix

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad (41)$$

we can rewrite $\bar{a}_{I\mu}$ as $\bar{a}_{I\mu} = U^T \tilde{a}_{I\mu}$, and rewrite Eqn. (39) as

$$\mathcal{L} = \frac{1}{4\pi} \tilde{K}_{IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} \tilde{q}_I A_\mu^{\text{spin}} \partial_\nu \tilde{a}_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (42)$$

where

$$\tilde{K} = U \bar{K} U^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{q} = U \bar{q} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}. \quad (43)$$

Eqn. (42) is the low energy effective theory for the spin-1/2 boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. We see that the mode $\tilde{a}_{3\mu}$ is gapless. Such a gapless mode corresponds to the total density fluctuations of the spin-1/2 bosons, indicating that the bosons are in a superfluid phase. But this is an unusual superfluid phase where the spin degrees of freedom form an SU(2) SPT phase.

To see this point, let us assume that the unit cell is large enough so that there are, on average, two spin-1/2 bosons per unit cell. In this case, when the repulsion between the bosons is large enough, the bosons may form a Mott insulator state. Such a Mott insulator state is described by the confinement of $\tilde{a}_{3\mu}$ U(1) gauge field.

So we can drop the $\tilde{a}_{3\mu}$ U(1) gauge field and obtain the following low energy effective theory in the Mott insulator phase:

$$\mathcal{L} = \frac{1}{4\pi} \tilde{K}_{2,IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} \tilde{q}_{2,I} A_\mu^{\text{spin}} \partial_\nu \tilde{a}_{I\lambda} \epsilon^{\mu\nu\lambda} \quad (44)$$

where

$$\tilde{K}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{q}_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}. \quad (45)$$

Since $\det(\tilde{K}_2) = -1$, the state $|\Phi\rangle$ (in the Mott insulator phase) is a SPT state. So, such a low energy effective theory describes a SU(2) SPT state, which has a spin Hall conductance for the S_z “charge”

$$\sigma_{xy} = \tilde{q}_2^T \tilde{K}_2^{-1} \tilde{q}_2 \frac{1}{2\pi} = 2 \frac{1}{4} \frac{1}{2\pi}. \quad (46)$$

We note that the spin Hall conductance is an even integer in the unit of $\frac{1}{4} \frac{1}{2\pi}$. Although the above discussion is for spin-1/2 bosons, a similar construction can be done for more physical spin-1 bosons, which give us a SO(3) SPT state.

V. CONCLUSION

In conclusion, a general approach is proposed to construct bosonic SPT phases on 2D lattice with various

symmetries. Our approach is based on the projective construction where the bosons/spins are fractionalized into several fermions which occupy nontrivial Chern bands. The bosonic SPT phases then can be constructed from the fermion state by Gutzwiller projection. We can calculate the low energy effective Chern-Simons theory of the projected states, which allows us to determine what kinds of SPT states are obtained after the Gutzwiller projection. We have constructed a U(1) SPT state, a SO(3) SPT state, and a SU(2) SPT state for spin-1 and spin-1/2 bosons.

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